Convex optimization in quantum information

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based on joint work with M. Berta (Caltech) & O. Fawzi (ENS Lyon)

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GHENT UNIVERSITY

Convex optimization and information theory

- Many problems in (classical) information theory can be formulated as convex optimization problems
- Examples: Computing capacities, determining good coding schemes, finding upper bounds on errors (to be discussed),…
- There has already been a very fruitful relationship between classical information theory and optimization theory
- **Quantum information theory**: in very abstract terms, **commuting** variables have to be replaced by **non-commuting** variables -> optimization over non-commutative fields
- **This talk**: an introduction into the the subject, based on the problem of sending information over a noisy channel
- In particular, we will focus on constructing **relaxations** to obtain **upper bounds**

Overview

- Classical noisy channel coding
- Linear program relaxations and the meta-converse
- Entanglement-assisted *channel coding*
  (**quantum assistance**)
- Semidefinite programming (SDPs)
- SDP hierarchies for understanding (bounding) the difference between classical and quantum variables
- Conclusion / Outlook

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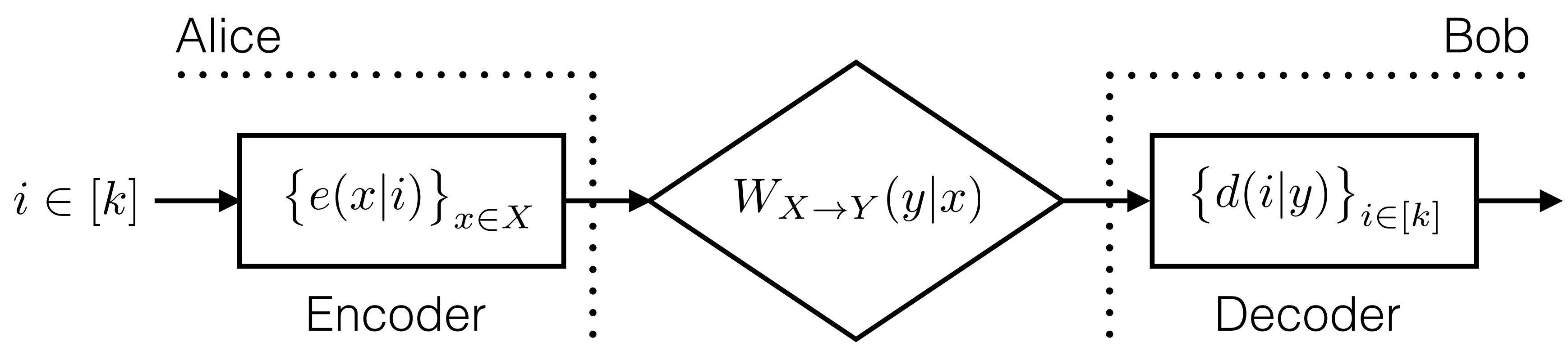
Classical noisy channel coding (I)



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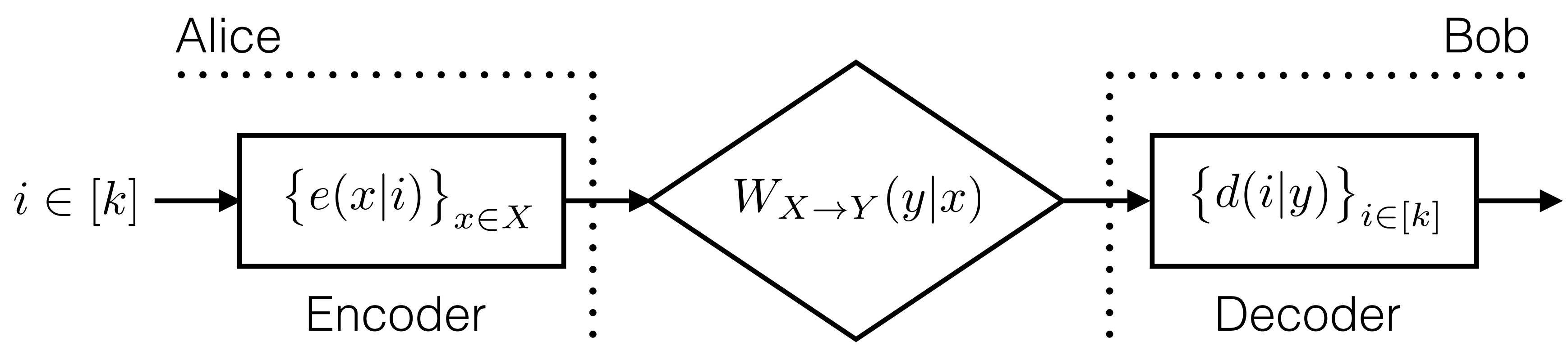

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$$W_{X \to Y}(y|x) \; \forall (x, y) \in X \times Y$$

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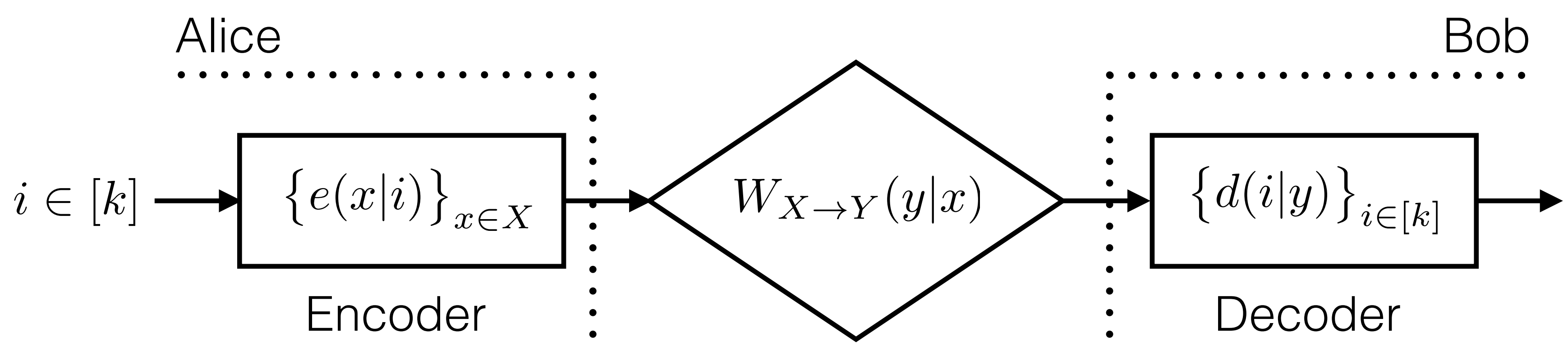

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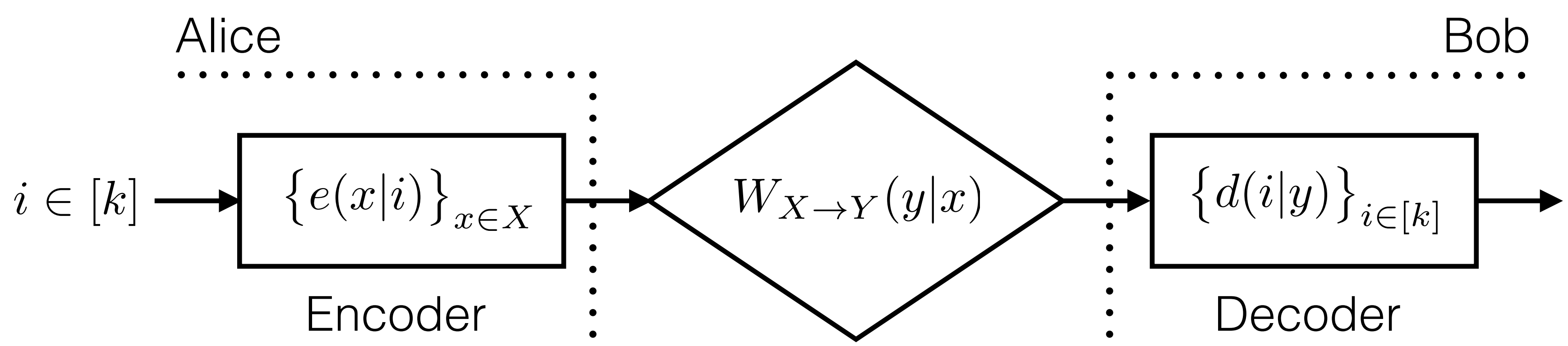

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"bilinear optimisation"

$$\begin{aligned} p_{\mathrm{succ}}(W,k) := \underset{(e,d)}{\text{maximize}} \quad & \frac{1}{k}\sum_{x,y,i} W_{X\to Y}(y|x)e(x|i)d(i|y) \\ \text{subject to} \quad & \sum_x e(x|i) = 1 \quad \forall i \in [k]\,, \quad \sum_i d(i|y) = 1 \quad \forall y \in Y \\ & 0 \le e(x|i) \le 1 \quad \forall (x,i) \in X \times [k]\,, \quad 0 \le d(i|y) \le 1 \quad \forall (i,y) \in [k] \times Y\,. \end{aligned}$$

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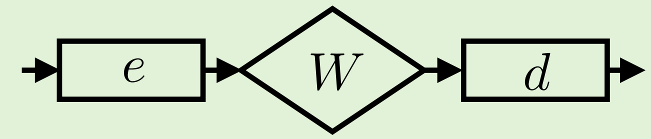
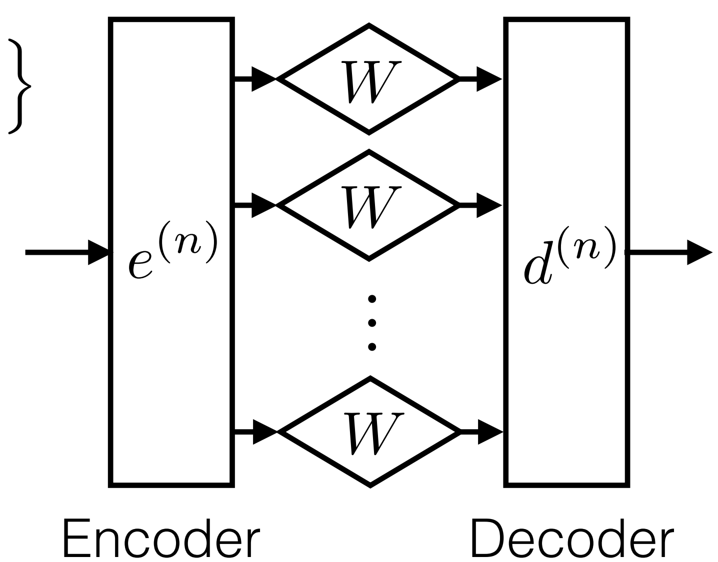


compared to

- Shannon's asymptotic independent and identical distributed (iid) channel capacity:

Definition: $C(W) := \sup\left\{R \,\middle|\, \forall\delta > 0 : \lim_{n\to\infty} p_{\mathrm{succ}}\left(W^{\times n}, [R(1-\delta)]^n\right) = 1\right\}$

Answer: $C(W) = \max_{P_X} I(X:Y)$ mutual information



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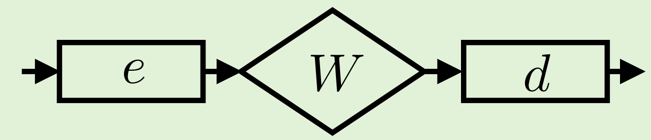

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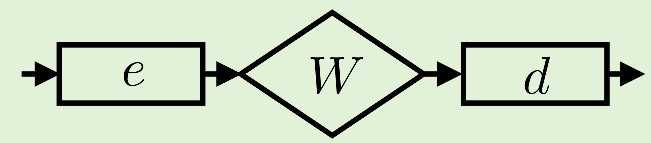

idea: consider this quantity as a matrix in x and y variables, and collect all its linear constraints

$$r(x,y) = \sum_i e(x|i)d(i|y)$$

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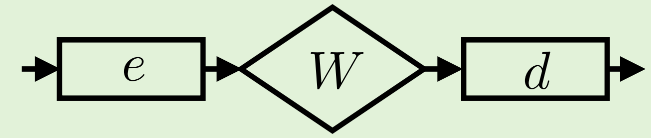

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Then:

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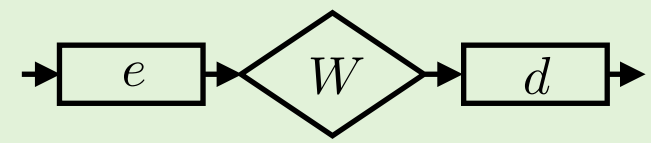

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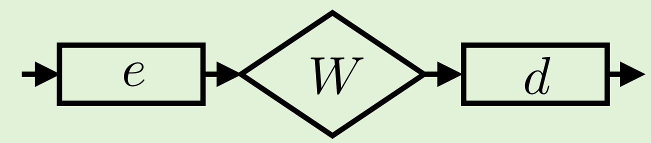

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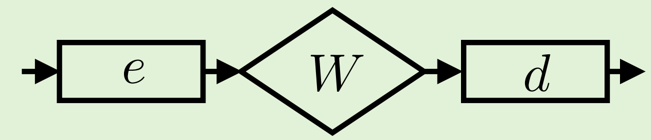

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And also: $$r(x,y) = \sum_i e(x|i)d(i|y) \le \sum_i e(x|i) = p(x)$$

Linear program relaxations (II)

- The bilinear program can be relaxed to a linear program:

$$
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\underset{(r,p)}{\text{maximize}} \quad & \frac{1}{k} \sum_{x,y} W_{X\to Y}(y|x) r(x,y) \\
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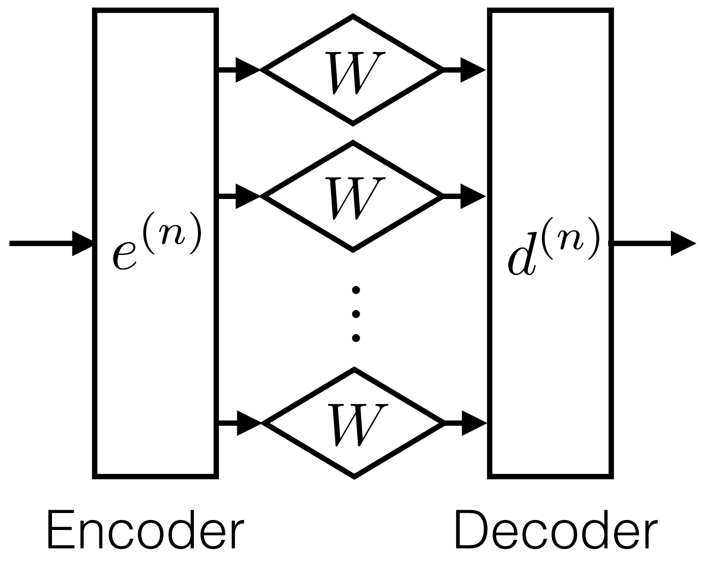

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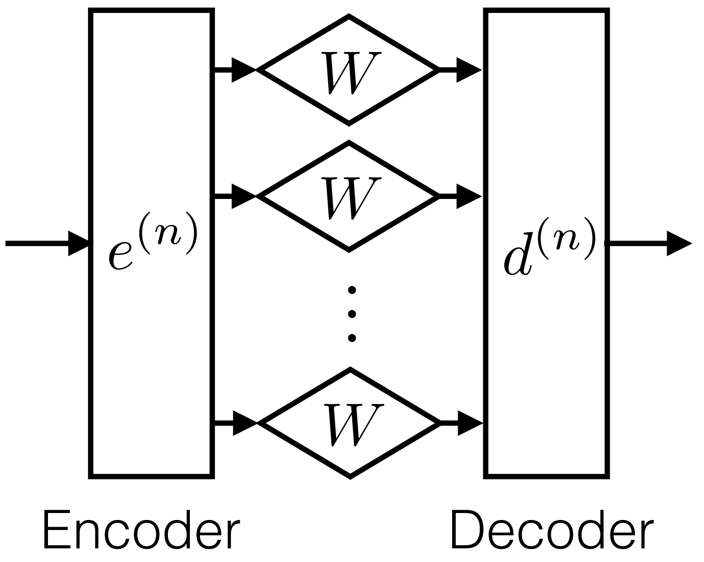
- This is the meta-converse of Polyanski and Verdu (c.f. Matthews)
- It converges to the exact value of the bilinear optimization program in the limit of many channel uses
- Can be used to obtain very sharp finite n bounds



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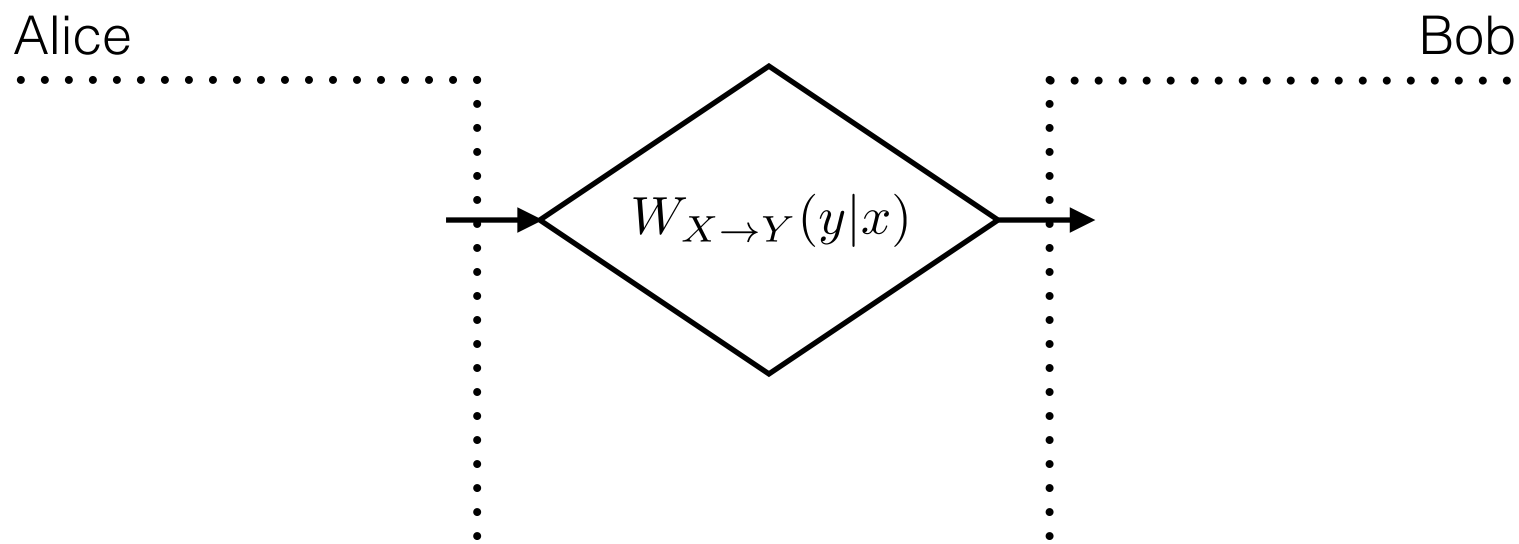
Adding quantum assistance….
but first recap what “quantum” means

- Classical variables are replaced by operators (“matrices”) on a Hilbert space $\mathcal{H}$
- Hilbert space $\mathcal{H}$ = linear vector space equipped with a scalar product (sesquilinear positive definite form)
- Configurations of the physical system are described by elements $|\psi\rangle$ (“states”) of unit norm of the Hilbert space
- Separated labaratories are described by taking the tensor product of their respective Hilbert spaces $\mathcal{H} \otimes \mathcal{H}$

Entanglement-assisted channel coding (I)



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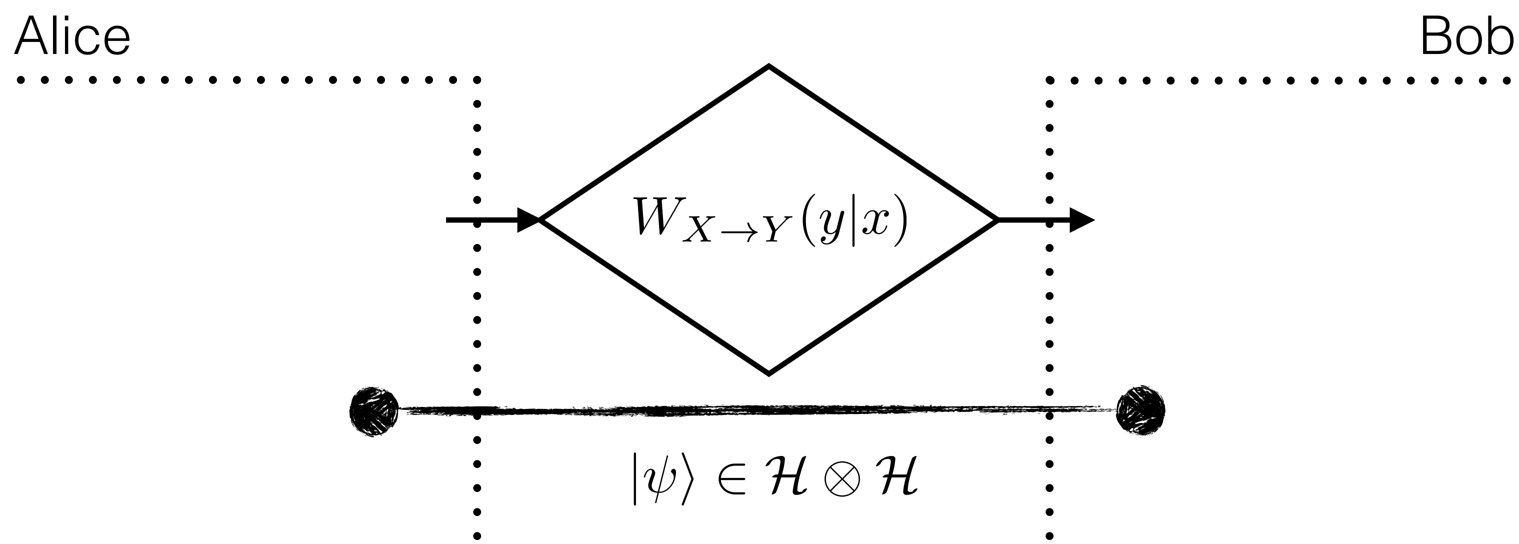

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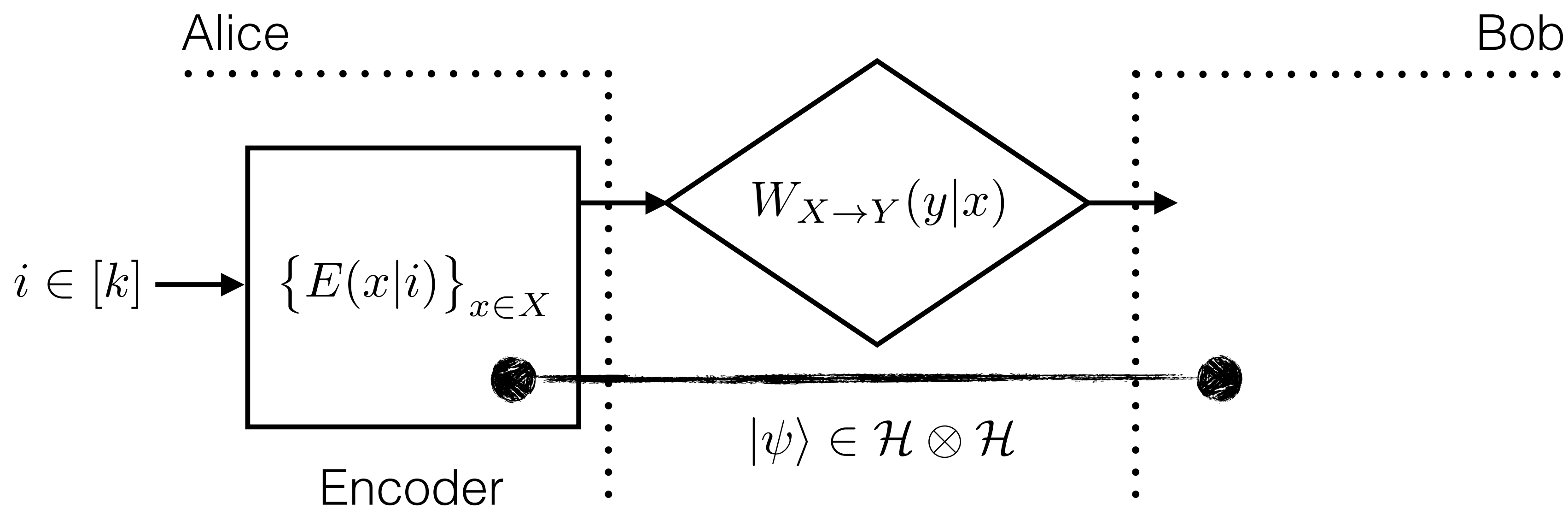

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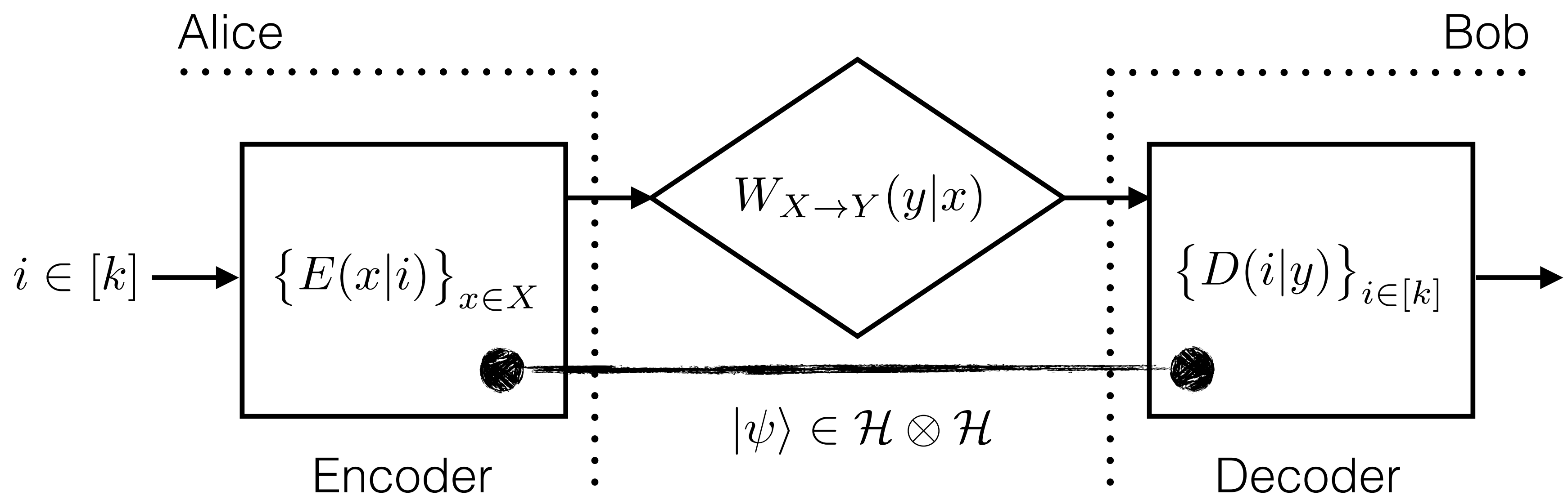

- The encoder and the decoder share an entangled quantum state
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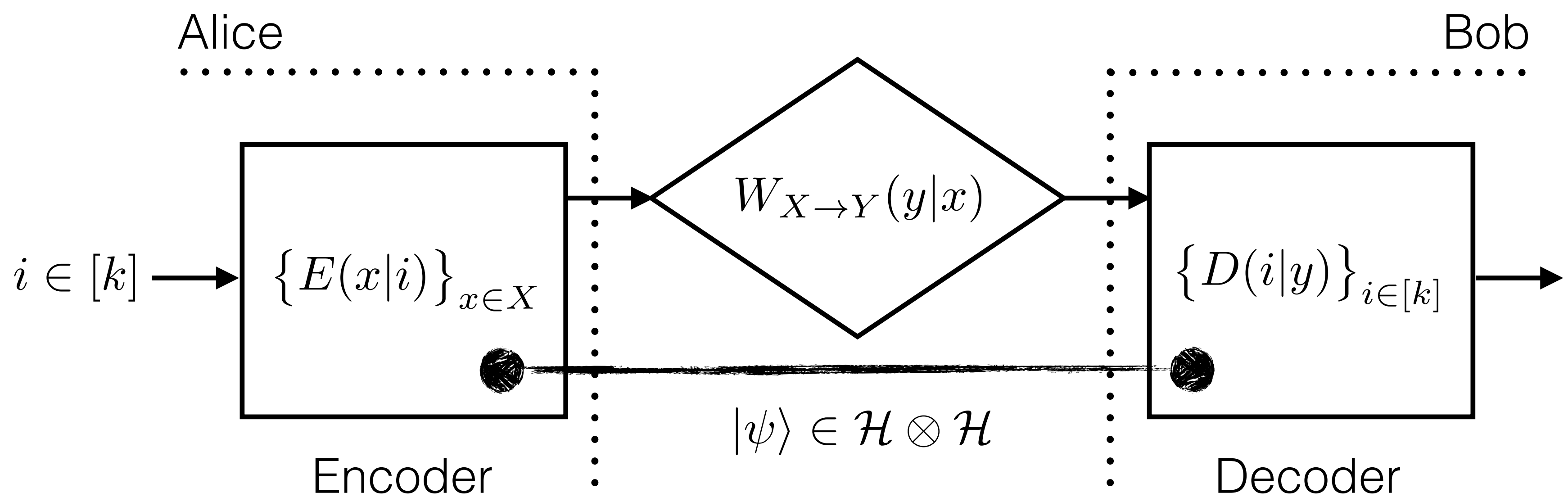

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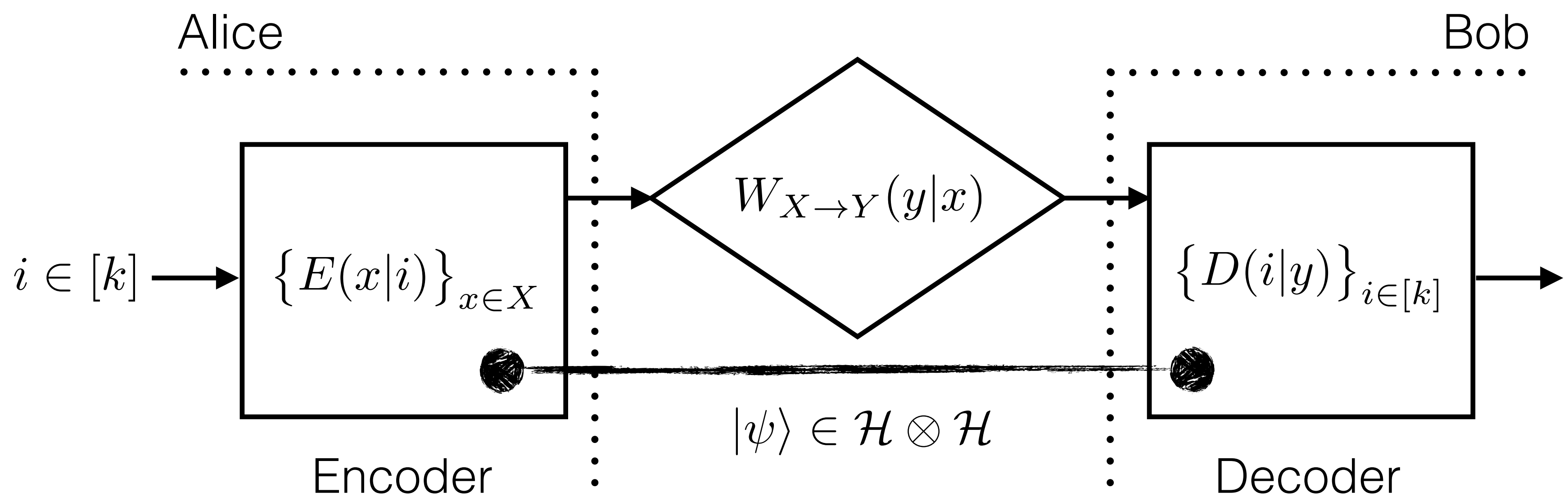

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- **Scalar** (commutative) versus **matrix** (non-commutative) variables:

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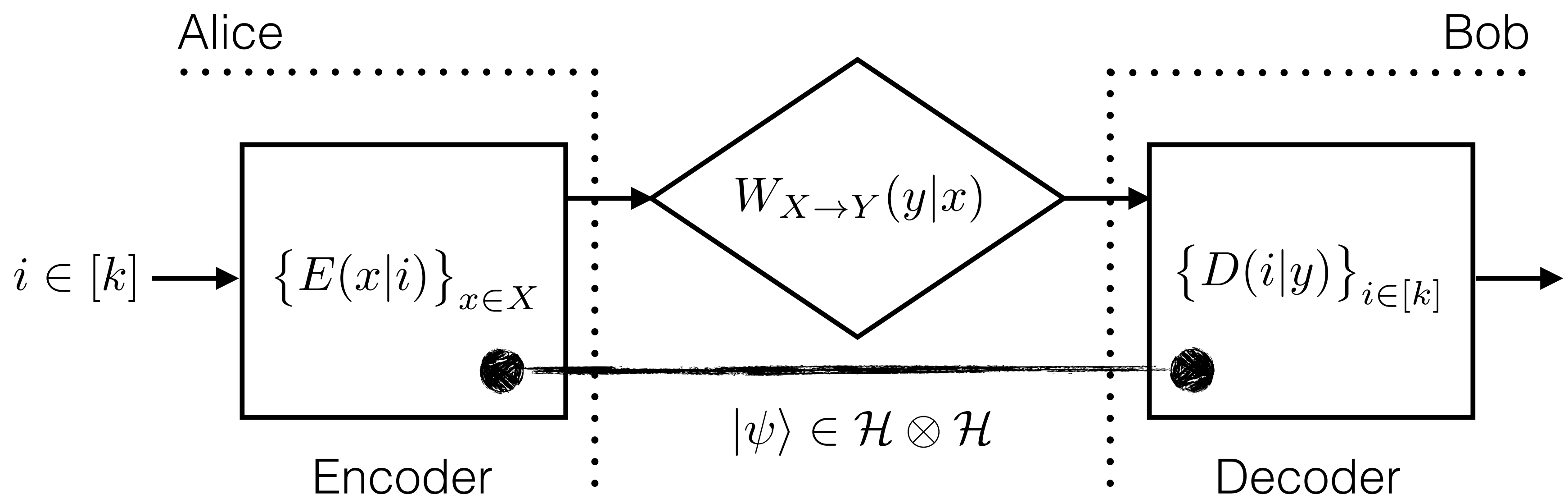

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- Unknown if $p^*_{\mathrm{succ}}(W,k)$ is computable!

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- Understand the possible separation: $p_{\mathrm{succ}}(W, k) \quad \text{versus} \quad p^{*}_{\mathrm{succ}}(W, k)$

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- In general, there is a **separation**:

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$$p_{\text{succ}}(Z,2) = \frac{5}{6} \approx 0.833 \quad \text{vs.} \quad p^*_{\text{succ}}(Z,2) \geq \frac{2+2^{-1/2}}{3} \approx 0.902$$

[Prevedel et al., PRL (2011)]

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[Prevedel et al., PRL (2011)]

—> this is also optimal with two-dimensional assistance

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[Williams and Bourdon, arXiv:1109.1029]

Entanglement-assisted channel coding (II)

- Understand the possible separation: $p_{\text{succ}}(W,k) \quad \text{versus} \quad p^*_{\text{succ}}(W,k)$

- ~~For the asymptotic iid capacity entanglement (quantum) assistance does not help:~~
  ~~$C(W) = C^*(W)$ [Bennett et al., PRL (1999)]~~

- In general, there is a **separation**:

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How to obtain upper bounds for the entangled scenario?

Overview

- Classical noisy channel coding
- Linear program relaxations and the meta-converse
- Entanglement-assisted *channel coding* (**quantum assistance**)
- Semidefinite programming (SDPs)
- SDP hierarchies for understanding (bounding) the difference between classical and quantum variables
- Conclusion / Outlook

Semi-definite programming (SDPs) (I)

- A semi-definite program (SDP) is a triple (T, A, B) with T a hermitian preserving linear map on matrices, and A, B hermitian matrices

- We can associate to such a triple two optimization programs

$$\begin{aligned} \underset{X}{\text{maximize}} \quad & \mathrm{Trace}[X\,A] \\ \text{subject to} \quad & X \geq 0 \\ & T(X) \leq B \end{aligned} \qquad\qquad \begin{aligned} \underset{X}{\text{minimize}} \quad & \mathrm{Trace}[Y\,B] \\ \text{subject to} \quad & Y \geq 0 \\ & T(X) \geq A \end{aligned}$$

- Most often, their value agree; this leads to efficient (in terms of the size of the matrices and the approximation error) optimization algorithms

- SDPs can be used to obtain a hierarchy of outer approximations to convex optimization problems (Lassere and Parillo)

Semi-definite programming (SDPs) (II)

- SDPs can be used to obtain a hierarchy of outer approximations to convex optimization problems (Lassere and Parillo)

- Idea: finding a consistent way to construct SDP relaxations of convex optimization problems

- Can be generalized to non-commutative variables (Navascues et. al., Doherty et. al.)

- Our contribution: **new converging sequence of tighter SDP relaxations** for **quantum bilinear optimization** problems such as the channel coding problem with entanglement assistance

$p_{\mathrm{succ}}(W,k) \leq p^*_{\mathrm{succ}}(W,k) = \mathrm{sdp}_\infty(W,k) \leq \ldots \leq \mathrm{sdp}_1(W,k)$ <— *efficiently computable!*

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First level semidefinite programming relaxation (I)

- Quantum bilinear program:

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p^*_{\mathrm{succ}}(W,k) := \underset{(\mathcal{H},\psi,E,D)}{\text{maximize}} \quad & \frac{1}{k}\sum_{x,y,i} W_{X\to Y}(y|x)\langle\psi|E(x|i)\otimes D(i|y)|\psi\rangle \\
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*motivated by: "**NPA hierarchy**" (Bell inequalities)*

[Lasserre, SIAM (2001)], [Parrilo, Math. Program. (2003)], [Navascues et al., PRL (2007)], [Doherty et al., IEEE CCC (2008)], [Navascues et al., NJP (2008)], [Pironio et al., SIAM (2010)]

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- First step: see as the part of the upper-right block of the Gram matrix

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- Original constraints can be formulated as positivity conditions on $\Omega:\ \mathrm{sdp}_1(W,k)$

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\text{subject to} \quad & \Omega \in \text{Pos}(1+k|X|+k|Y|)\,, \quad \Omega_{\emptyset,\emptyset} = 1 \quad \text{with } \emptyset \text{ the empty symbol} \\
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- Going back to our example:

$$p_{\text{succ}}(Z,2) = \frac{5}{6} \approx 0.833$$

(known before, with two-dimensional assistance)

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- Relaxation: $p^*_{\mathrm{succ}}(Z,2) \leq \mathrm{sdp}_1(Z,2) \approx 0.908 = \frac{1}{2} + \frac{1}{\sqrt{6}}$
- Four-dimensional assistance: $p^*_{\mathrm{succ}}(Z,2) \geq \frac{1}{2} + \frac{1}{\sqrt{6}}$

Conclusions

- Fruitful interplay of optimization theory and (quantum) information theory
- Going quantum (= adding quantum assistance to classical tasks) roughly means that commutative variables have to replaced by non-commutative ones
- Optimization becomes harder, but semi-definite programming can be used to obtain converging upper bounds
- Many more directions to explore: fully quantum scenario, other information theoretic protocolls,…

Thanks!